Local Prescribed Mean Curvature foliations in cosmological spacetimes

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Abstract

A theorem about local in time existence of spacelike foliations with prescribed mean curvature in cosmological spacetimes will be proved. The time function of the foliation is geometrically defined and fixes the diffeomorphism invariance inherent in general foliations of spacetimes. Moreover, in contrast to the situation of the more special constant mean curvature foliations, which play an important role in global analysis of spacetimes, this theorem overcomes the existence problem arising from topological restrictions for surfaces of constant mean curvature.

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1 Introduction

Spacelike foliations arise in many contexts in General Relativity. This is related to the fact, that the world around us seems three dimensional and we are used to think about dynamics as a sequence of processes parametrized by some time function. But in Relativity there is no canonical time coordinate, thus any time function only fixes (a part of) the diffeomorphism invariance of the theory but this gauge fixing is a priori arbitrary unless the time function is tied to some geometrically defined quantity. For that reason, an arbitrary time function does not provide any information about the global structure of the spacetime under consideration. For example, any thin, spacelike strip in Minkowski spacetime can be regarded as global in time, by stretching the time coordinate with an appropriate diffeomorphism. This problem is connected with the lack of a canonical metric background structure for Einstein's field equations.

In this work I will define and prove a uniqueness and local in time existence theorem for a spatially global, geometrically defined time coordinate in cosmological spacetimes in terms of a *Prescribed Mean Curvature (PMC)* foliation, defined as follows: The leaves $\{S_t\}$ are determined implicitly by the requirement, that given an initial Cauchy surface $S_0 := \Sigma$ subject to a certain condition, the mean curvature at each point $p_t \in S_t$ is defined by the relation $H(p_t) = H(p_0) + t$, where p_t and $p_0 \in S_0$ are connected by the integral curves $\gamma(t)$ of the normal vector field of the leaves. Therefore, the elements in this construction are purely geometrical in nature, the time function obtained by this procedure depends only on the choice of the initial Cauchy surface and we get a geometrical measure of the (timelike) size of its Cauchy development as long as this time coordinate exists. I will refer to the leaves of such a foliation as surfaces of prescribed mean curvature. Note, that there is another definition for surfaces with prescribed mean curvature in the literature ('H-surfaces'), compare [G]. But unlike the definition given there, where the spacetime and a function H prescribing the mean curvature are given quantities, the definition here relates to the PMC foliation as a whole, providing an intrinsic prescription without having given the spacetime or a function H in advance: Coupling the PMC equations to Einstein's evolution equations with matter (such that the Cauchy problem in harmonic coordinates is well posed), one gets a system of equations, which constructs a spacetime foliated in this way, where the mean curvature 'develops' from leaf to leaf.

Given this kind of foliation globally, we have a tool at hand for the global analysis of spacetimes in an invariant manner, in terms of an asymptotic analysis of a system of partial differential equations, throwing away the differential geometric difficulties arising from the diffeomorphism invariance. For an introduction to global issues in Relativity compare the survey article [An]. Related topics are the development of singularities, the cosmic censorhip conjecture and the closed universe recollapse conjecture. Introductions to these topics can be found in [ME] (global existence and cosmic censorship) and [BT],[BGT] (closed universes and recollapse).

A related construction are the well-known constant mean curvature (CMC) foliations, see the survey article [R] or [MT] for reference. Obviously they are a special case of a PMC foliation, when the initial Cauchy surface itself has constant mean curvature, and this fact sheds some light about the origins of PMC foliations. Indeed, the need for such a generalization arises from the fact, that there are strong topological restrictions concerning the existence of even a single CMC hypersurface. There exist spacetimes, which do not possess any constant mean curvature hypersurface at all, see [Ba]. Conequently, many of the results obtained by constructions with CMC foliations presuppose the existence of

at least a single CMC hypersurface, which then has to be verified in each concrete case. Contrary to this, for the existence of a local in time PMC foliation there is a simple and easily verified criterion (compare theorem 4.1), leading to a broad class of spacetimes, possessing at least a local in time PMC foliation. The similarity between PMC and CMC foliations then gives rise to the hope, that the global results obtained in the CMC cases (under the additional existence assumptions mentioned above) can also be proved using PMC foliations, an issue a forthcoming paper will be concerned with.

The organisation of this work is as follows. The section 2 fixes notation and introduces the basic equations and results used throughout this work. Its first part is concerned with General Relativity, while the second part is devoted to partial differential equations. Section 3 then contains the construction of the local in time PMC foliation. In the last section the main result will be stated and discussed.

2 Preliminaries

2.1 Spacetimes and foliations

A spacetime is a pair (M, g), where M denotes a four dimensional smooth and orientable Lorentz manifold with metric g and signature $(-+++)$. The metric induces structures like the Levi-Civita connection ${}^4\nabla$ and the curvature on M. The sign convention for the curvature is fixed by the definition ${}^4R(X,Y)Z := {}^4\nabla_X {}^4\nabla_Y Z - {}^4\nabla_Y {}^4\nabla_X Z - {}^4\nabla_{[X,Y]} Z$, where X, Y Z are vector fields. The curvature tensor is then defined as ${}^4R(W,X,Y,Z) := g(W, {}^4R(X,Y)Z)$ with Ricci tensor ${}^4R_{\alpha\beta} = {}^4R^{\mu}_{\alpha\mu\beta}$ and scalar curvature ${}^4R = {}^4R^{\mu}_{\mu}$, written in abstract index notation of the Ricci calculus. The Einstein tensor reads $G_{\alpha\beta} = {}^4R_{\alpha\beta} - \frac{1}{2}{}^4R g_{\alpha\beta}$ and we can write Einstein's field equation as

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{1a}$$

or equivalently (by ${}^4R = -8\pi \operatorname{tr} T$)

$${}^4R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \tfrac{1}{2}(\operatorname{tr} T)\, g_{\alpha\beta} \right) \quad , \tag{1b}$$

where $T_{\alpha\beta}$ denotes the energy momentum tensor of the matter fields. It is a symmetric tensor on M with vanishing divergence as a consequence of the Bianchi identities, a requirement which imposes supplementary conditions on the matter fields coupled to the field equation.

The components of the energy momentum tensor with respect to an observer, represented by a unit timelike vector n, have physical meaning: Denote by $h_{\alpha\beta} := g_{\alpha\beta} + n_\alpha n_\beta$ the orthogonal projector on $\{n\}^\perp$ in covariant notation, then we can define the energy density, momentum density and the stress tensor by

$$\rho := T_{\mu\nu} n^\mu n^\nu \tag{2a}$$

$$j_\beta := -T_{\mu\nu} n^\mu h^\nu_\beta \tag{2b}$$

$$S_{\alpha\beta} := T_{\mu\nu} h^\mu_\alpha h^\nu_\beta \quad . \tag{2c}$$

In this work we confine ourselves to *cosmological* solutions (M, g) of Einstein's field equations. Due to [Ba] these are globally hyperbolic and spatially compact spacetimes, where the Ricci tensor contracted twice with any timelike vector is non-negative (timelike convergence condition). This last condition can be reexpressed in terms of the matter variables as $\rho + \operatorname{tr} S \geq 0$ for any observer, known as the strong energy condition.

An introduction to General Relativity with a treatment of the Cauchy problem for the field equations is [W1], while a recent presentation with a deeper analysis can be found in [FR].

Now let us pay attention to an additional structure. A foliation $\{S_t\}$, $t \in I \subset \mathbb{R}$ (I interval containing zero) of (a part of) (M, g) by spacelike hypersurfaces induces on each leaf the unit normal vector field n, the metric $h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$, which also serves as orthogonal projection and the second fundamental form $k_{\alpha\beta} := -h^\mu_\alpha h^\nu_\beta {}^4\nabla_\mu n_\nu$ (the definition of $k_{\alpha\beta}$ fixes the sign conventions used in this work). The second fundamental form is a symmetric tensor, intrinsic to the leaves of the foliation, and it can also be written as the Lie derivative of the 3-metric h with respect to the normal vector field, $k_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_n h_{\alpha\beta}$. The 3-metric determines further geometrical objects on the leaves, such as the Levi-Civita connection ∇ and the curvature tensor $R(\cdot)$. Tensors intrinsic to the leaves of the foliation

will carry Latin indices in the abstract index notation.
The parameter t of the foliation has timelike gradient and thus can be regarded as (coordinate-) time. Given local coordinates (x^i) on S_0, we can Lie-transport them to neighbouring leaves along an arbitrary family of transversal curves, parametrized by x. We will express equations containing coordinate components with respect to the adapted coordinates (t, x).
We define the lapse function N and the shift vector $\nu \perp n$ on the leaves by the formula

$$(3) \qquad \partial_t = Nn + \nu \quad \text{, thus} \quad \begin{aligned} N &= -g(\partial_t, n) \\ \nu &= \partial_t - Nn \end{aligned}$$

Then we find $1 = dt(\partial_t) = N\,dt(n)$. Further, dt is (co-)orthogonal on the leaves and if we denote the conormal of the leaves by σ we see that $dt = -N^{-1}\sigma$ or $\sigma = -Ndt$. Thus N^{-1} measures the length of dt, which can be interpreted as follows: For an observer along n N measures the elapsed proper time from S_t to S_{t+dt} or equivalently, the elapsed coordinate time dt along a journey along n between two leaves, separated by a unit distance (measured along n) is N^{-1}, thus N^{-1} measures the elapsed coordinate time along n. Note that these relations have only infinitesimal meaning and for the elapsed proper time along n between S_{t_1} and S_{t_2} we have $\int_{t_1}^{t_2} N$.

The coordinates adapted to the foliation together with lapse and shift fix the diffeomorphism invariance and we get a 3+1-split of the field equations consisting of constraint equations intrinsic to the leaves and evolution equations.
The constraint equations are a consequence of the equations of Gauss and Codazzi, where Einstein's equations have been used to eliminate the curvature terms of the 4-geometry. The result is

$$(4a) \qquad R + H^2 - |k|^2 = 16\pi\rho \qquad \text{(Hamiltonian constraint)}$$

$$(4b) \qquad \nabla^j k_{ij} - \nabla_i H = 8\pi j_i \qquad \text{(momentum constraint)} \quad ,$$

with $|k|^2 = k_{\alpha\beta}k^{\alpha\beta}$ and $H = \operatorname{tr} k$ denotes the mean curvature of the leaves.
The remaining evolution equations for the components of the first and second fundamental forms are known as ADM equations. They read in adapted coordinates

$$(5a) \qquad \partial_t h_{ij} = -2Nk_{ij} + \nabla_i \nu_j + \nabla_j \nu_i$$

$$(5b) \qquad \begin{aligned} \partial_t k_{ij} = &-\nabla_i \nabla_j N + N\Big(R_{ij} + Hk_{ij} - 2k_i^r k_{rj} - 8\pi\big(\,S_{ij} + \tfrac{1}{2}(\rho - \operatorname{tr} S)h_{ij}\,\big)\Big) \\ &+\nu^r \nabla_r k_{ij} + k_{rj}\nabla_i \nu^r + k_{ir}\nabla_j \nu^r \quad , \end{aligned}$$

where the first equation is merely a rewriting of the definition of the second fundamental form, $k = -\frac{1}{2}\mathcal{L}_n h$. Taking the trace of the second equation and eliminating the scalar curvature R by the Hamiltonian constraint we obtain the lapse equation

$$(6) \qquad \Delta N + N\Big(|k|^2 + 4\pi(\,\rho + \operatorname{tr} S\,)\Big) = (\partial_t - \nu)H \quad ,$$

which serves as a constraint of the foliation, induced by Einstein's equations. Note, that in cosmological spacetimes, the term in brackets is always non-negative.

2.2 Functional analysis and partial differential equations

At first I state some basic functional analytic definitions. Making use of the multi index notation, the Sobolev spaces are defined as

$$\begin{aligned} W^{k,p}(\mathbb{R}^n) &:= \{f \in L^p(\mathbb{R}^n) \mid \partial^\alpha f \in L^p(\mathbb{R}^n) \, \forall_\alpha |\alpha| \leq k\} \quad , \\ H^k(\mathbb{R}^n) &:= W^{k,2}(\mathbb{R}^n) \end{aligned}$$

and for $s \in \mathbb{R}$ we define

$$H^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) \mid (1+|\xi|^2)^{s/2} \mathcal{F} f \in L^2(\mathbb{R}^n)\} \quad ,$$

where $\mathcal{S}'$ denotes the space of tempered distributions and $\mathcal{F}$ is the Fourier transform. There are continuous embeddings $H^s(\mathbb{R}^n) \subset H^{s'}(\mathbb{R}^n)$ for $s' < s$, thus the elements of $H^s(\mathbb{R}^n)$, $s \geq 0$ are functions. Furthermore, to compare different Sobolev spaces I cite [Ra], (Appendix A) where some basic facts from the theory of interpolation spaces are stated:
For $s_0 \neq s_1$ in $\mathbb{R}$ let $s_\theta := (1-\theta)s_0 + \theta s_1$, $0 < \theta < 1$. Then

$$[H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n)]_\theta = H^{s_\theta}(\mathbb{R}^n) \quad ,$$

thus $H^{s_\theta}(\mathbb{R}^n)$ is an intermediate space between $H^{s_0}(\mathbb{R}^n)$ and $H^{s_1}(\mathbb{R}^n)$.

For the convenience of the reader I now list some (well known) inequalities, that are central to the analysis of partial differential equations. I adopt the convention of denoting any generic constant by C.

The perhaps most important estimate, we will make use of very often is Gronwall's inequality.

2.1. Proposition (Gronwall's inequality).
Let $I \subset \mathbb{R}$ be an interval, $t_0 \in I$ and $\alpha, \beta, u \in C(I, \mathbb{R}_+)$, with

$$u(t) \leq \alpha(t) + \left| \int_{t_0}^t \beta(s) u(s) \, ds \right|$$

for all $t \in I$.
Then

$$u(t) \leq \alpha(t) + \left| \int_{t_0}^t \alpha(s)\beta(s) \, e^{\left|\int_s^t \beta(r) \, dr\right|} \, ds \right|$$

holds for all $t \in I$.

The proof of Gronwall's inequality in this particular form can be found in [A].

The second important estimate, relating differentiability classes to each other is the collection of the two Sobolev inequalities, which can be found in standard textbooks about partial differential equations or functional analysis, see for example [RR]. Here, D^s abbreviates the vector with components D^α for all α with $s = |\alpha|$.

2.2. Proposition (Sobolev inequalities).

(S1) $1 \leq kp < n$

$$\begin{aligned} &W^{k,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-kp}}(\mathbb{R}^n) \\ &\|u\|_{L^{\frac{np}{n-kp}}} \leq C \|u\|_{W^{k,p}} \end{aligned}$$

(S2) $kp > n$

$$W^{k,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$$
$$\|u\|_{L^\infty} \leq C\|u\|_{W^{k,p}}$$

In particular we will use a special case of the second inequality,

2.3. Corollary (Sobolev embedding theorem).

$k > \frac{n}{2} + l$

$$H^k(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \cap C^l(\mathbb{R}^n)$$
$$\|u\|_{C^l} \leq C\|u\|_{H^k}$$

For the analysis of non-linear equations the Moser estimates are essential

2.4. Proposition (Moser inequalities).

(M1) $f, g \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),\ s = |\alpha|$

$$\|D^\alpha(fg)\|_{L^2} \leq C\left(\|f\|_{L^\infty}\|D^s g\|_{L^2} + \|D^s f\|_{L^2}\|g\|_{L^\infty}\right)$$

(M2) $f \in H^s(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n),\ g \in H^{s-1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),\ s = |\alpha|$

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2} \leq C\left(\|D^s f\|_{L^2}\|g\|_{L^\infty} + \|Df\|_{L^\infty}\|D^{s-1}g\|_{L^2}\right)$$

(M3) $f \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),\ F \in C^\infty(\mathbb{R}),\ F(0) = 0,\ B_R := \{x \in \mathbb{R}^n \mid \|x\| \leq R\}$

$$\|D^s F(f)\|_{L^2} \leq c\left(\|f\|_{L^\infty}\right)\|D^s f\|_{L^2} \quad ,$$

with $c(\|f\|_{L^\infty}) = C\|F\|_{C^s(B_{\|f\|_{L^\infty}})}$

(see for example [Ra]).

Let us turn now to the analysis of special classes of partial differential equations. We first consider hyperbolic equations:

2.5. Definition (Symmetric hyperbolic system).
Let $\{0\} \subset I \subset \mathbb{R}$ denote an interval and $G \subset \mathbb{R}^k$ an open set. Further, for $i = 1, \ldots, n$ let A^0, A^i, B denote smooth functions on $I \times \mathbb{R}^n \times G$, where $A^0 - \mathrm{id}$, A^i and B have compact support. A^0, A^i are assumed to take values in $\mathbb{R}^{k\times k}$ and B in $\mathbb{R}^k$. Then we call the system

$$A^0(t,x,u)\,\partial_t u + A^i(t,x,u)\,\partial_i u + B(t,x,u) = 0$$

for the unknown $u : I \times G \longrightarrow \mathbb{R}^k$ *(quasilinear) symmetric hyperbolic*, if the matrices A^0, A^i are symmetric and A^0 is uniformly positive definite.
The system is called *(inhomogeneous) linear symmetric hyperbolic*, if it has the form

$$A^0(t,x)\,\partial_t u + A^i(t,x)\,\partial_i u + B(t,x)u = f(t,x) \quad .$$

The following proposition holds (see e.g. [Ra] for a proof):

2.6. Proposition (Existence and Uniqueness of solutions).

1. *Let $u_0 \in C_0^\infty(\mathbb{R}^n)$. Then there exists a unique solution $u \in C^\infty([0,\infty[\times\mathbb{R}^n)$ of the linear symmetric hyperbolic system, with $u(0,x) = u_0(x)$. Moreover, u satisfies the energy estimate*

$$\|u(t)\|^2_{H^s} \leq C\left(\|u(0)\|^2_{H^s} + \sup_{0\leq t'\leq t}\|f(t')\|^2_{H^s}\right)$$

2. *Let $s > \frac{n}{2}+1$, $u_0 \in H^s(\mathbb{R}^n)$ with compact support. Then there is a $T > 0$ which depends on $\|u_0\|_{H^s}$ and a unique classical solution $u \in C^1([0,T]\times\mathbb{R}^n)$ of the quasilinear symmetric hyperbolic system with $u(0,x) = u_0(x)$ and*

$$u \in \bigcap_{\mathbb{N}\ni k<s} C^k\left([0,T], H^{s-k}(\mathbb{R}^n)\right) \quad .$$

Moreover the energy estimate

$$1 + \|u(t)\|^2_{H^s} \leq \left(1 + \|u(0)\|^2_{H^s}\right)\, e^{C\int_0^t (1+\|u(t')\|_{C^1})\, dt'}$$

holds, providing some extension criterion if $\|u(t)\|_{C^1}$ remains bounded in $[0,T]$.

In contrast to hyperbolic equations the study of elliptic operators differs substantially from that of hyperbolic differential operators, because there is no finite propagation speed, not even propagation at all. Therefore we cannot localize and have to analyse elliptic equations as a whole instead.
In this work we need some standard facts about scalar linear elliptic equations on compact manifolds. Basics about elliptic operators can be found for example in [RR] and some more advanced results on Fredholm theory of elliptic operators on compact manifolds are presented in Cantor's article [C]. In particular we find for the Laplacian $\Delta := -h^{ij}\nabla_i\nabla_j$ of a smooth compact Riemannian manifold (Σ, h):

2.7. Proposition.
Let $k \geq 0$ be some integer. $\Delta : H^{k+2}(\Sigma) \longrightarrow H^k(\Sigma)$ is a Fredholm operator with $\mathrm{ind}(\Delta) = 0$, $H^k = \ker(\Delta) \oplus \mathrm{Im}(\Delta)$, *where the sum is L^2-orthogonal and* $\ker(\Delta) = \{$*constant functions*$\}$, *thus* $\mathrm{Im}(\Delta) = \{f \in C^\infty(\Sigma) \mid \int_\Sigma f = 0\}$.
Moreover, duality and interpolation shows, that Δ extends to a Fredholm operator $H^{s+2}(\Sigma) \longrightarrow H^s(\Sigma)$ of index zero for arbitrary $s \in \mathbb{R}$

The class of Fredholm operators is closed under addition of compact operators and the index remains constant. Thus we easily get the

2.8. Corollary.
If $\lambda \in C^\infty(\Sigma)$ is everywhere non-negative and does not vanish identically, then the elliptic operator

$$L = \Delta + \lambda : H^{s+2}(\Sigma) \longrightarrow H^s(\Sigma)$$

is an isomorphism for all $s \in \mathbb{R}$.

In particular, this corollary applies to the operator in (6) in cosmological spacetimes.

3 Local PMC foliations

3.1 The PMC equations

Let (M, g) be a smooth, globally hyperbolic spacetime with compact Cauchy surface Σ. We denote the coordinates on Σ by x^i and define U to be a Gaussian neighbourhood of the form $I \times \Sigma \supset]-\epsilon, \epsilon[\times \Sigma$, with some $\epsilon > 0$ of $\Sigma := \{t = 0\}$. Then we have $g(\partial_0, \partial_0) = -1, \partial_0 \perp \partial_i$, $i = 1, 2, 3$ and $\partial_0 := \partial_t$. D abbreviates the vector (∂_i) and for an intrinsic spacelike 3-vector v set $|v|^2 := g_{ab}v^a v^b$.

Let $0 \in J \subset \mathbb{R}$ denote an interval. A foliation $\{S_\tau\}$ of a neighbourhood $U' \subset U$ of Σ in M by spacelike hypersurfaces is given by a map $\varphi : J \longrightarrow C^\infty(\Sigma, \mathbb{R})$, $\tau \longmapsto \varphi_\tau$, with $|D\varphi_\tau| < 1$, $\tau \longmapsto \varphi_\tau(x)$ strictly monotone for all $x \in \Sigma$ and $S_\tau = \{(t, x) \,|\, \varphi_\tau(x) - t = 0\}$. The foliation defines a unit normal n, the first and second fundamental forms h and k respectively and the mean curvature H on each leaf, parametrized by τ. The coordinates on each leaf will be identified with the coordinates on Σ by the action of ϕ^*_τ, with $\phi_\tau(x) = (\varphi_\tau(x), x)$, so we can regard functions on the leaves as functions on Σ, parametrized by τ. With these conventions the foliation fulfills the equation

$$\text{(7a)} \qquad \frac{\partial \varphi}{\partial \tau} = \left(1 - |D\varphi|^2\right)^{\frac{1}{2}} N \quad ,$$

where we dropped the subscript τ from φ, now simply considered as a function of (τ, x). We fix the foliation by imposing a condition for the lapse function N, which has to obey

$$\text{(7b)} \qquad \begin{aligned} &L_\tau N := (\Delta_\tau + \lambda_\tau)\, N = 1 \quad , \\ &\Delta_\tau := -h^{\alpha\beta}\nabla_\alpha \nabla_\beta \quad , \\ &\lambda_\tau := k_{\alpha\beta}k^{\alpha\beta} + 4\pi(\rho + \operatorname{tr} S) \quad , \end{aligned}$$

which is the usual lapse equation (6) on S_τ (indicated by the subscript τ in the equations above), together with the *prescribed mean curvature condition* $Nn^\alpha \partial_\alpha H = 1$. Thus we force the mean curvature to grow along the normal vector field of the leaves as described in the introduction. We define a PMC foliation to be a solution of equations (7a), (7b) and our goal is to prove uniqueness and local in time existence of solutions of that system of equations.

To cast the hyperbolic part of the equations into first order symmetric hyperbolic form, we set $w := \binom{u}{v} := \binom{\varphi}{D\varphi}$ and express all functions in terms of w and N. Then one finds $n = n(w)$, $h = h(w)$, $k = k(w, Dw)$ and the equations read

$$\text{(8a)} \qquad \partial_\tau w + A^i(w, N)\, \partial_i w + B(w, N, DN) = 0$$

$$\text{(8b)} \qquad L_\tau(w, Dw)\, N = 1 \quad ,$$

with symmetric 4×4 matrices A^i and B takes values in $\mathbb{R}^4$.

Initial values are given by a function $w_0 \in H^s(\Sigma, \mathbb{R}^4)$, with $|v_0| < 1$, which represents a spacelike hypersurface of regularity H^s in M, with an appropriate choice of s. Since we want to have $S_0 = \Sigma$ we set $w_0 = 0$ throughout this work, but we do not make use of this explicit setting in the sequel. Rather we proceed in full generality assuming only $w_0 \in H^s(\Sigma, \mathbb{R}^4)$, spacelike.

In order to get a well-posed initial value problem for the system (8), we have to ensure, that $L_0(w_0, Dw_0)$ is an isomorphism of Sobolev spaces, so that (8b) has a unique solution

N_0 on w_0. In view of corollary 2.8 we require $\lambda_0 \geq 0$, $\lambda_0 \neq 0$ on w_0 which turns out to be satisfied for almost all Cauchy surfaces Σ in spacetimes obeying the strong energy condition $\rho + \operatorname{tr} S \geq 0$, while $\lambda \equiv 0$ would force Σ to be maximal and time symmetric. In the latter case a small deformation of Σ would suffice. Therefore, from now on we focus on cosmological spacetimes in the sense described in section 2.1.

The domain G of definition for the coefficients of the equations can be formally decomposed as $G = G_w \times G_{Dw} \times G_N \times G_{DN}$, where the factors are constraint by the requirements $|v| < 1$ in $\mathbb{R}^3$ and $N > 0$ in $\mathbb{R}$.

To prove local in time existence and uniqueness for solutions of the system (8), equivalent to (7), we invoke the following procedure: We define iteratively a sequence of uncoupled linear equations (see (9)), which have smooth solutions (w^j, N^j) on non-empty time intervals J^j. Then we have to show convergence $(w^j, N^j) \longrightarrow (w, N)$ in some appropriate Sobolev space on a non-empty time interval J. Finally we have to verify, that (w, N) represents indeed an unique solution of the system (8).

Let us start with the definition of the sequence mentioned above. Denote by (w_0^j) a sequence of smooth functions on Σ, with $|v_0^j| < 1$ and $w_0^j \longrightarrow w_0$ in $H^s(\Sigma)$. The iteration will be defined inductively:

0) Set

$$
\begin{aligned}
&w^0(\tau, x) := w_0^0(x) \quad \forall_{(\tau,x)\in I\times\Sigma} \\
&N^0 := 1 \quad \text{on} \quad I \times \Sigma \quad ,
\end{aligned}
$$

then one has on $J^0 := I$: $|v^0| = |v_0^0| < 1$, which establishes w^0 as a well-defined family of spacelike hypersurfaces in U. Moreover, $N^0 > 0$ by construction and $0 \neq \lambda^0(w^0, Dw^0) \geq 0$, by $0 \neq \lambda_0(w_0, Dw_0) \geq 0$ and continuity.

j) Now we have a non-empty time interval J^{j-1} for which a family of spacelike hypersurfaces w^{j-1} with $0 \neq \lambda^{j-1} \geq 0$ is defined, and $N^{j-1} > 0$ is given on J^{j-1}. Then one gets smooth solutions (w^j, N^j) on J^{j-1} for the (linear) system

$$
\begin{aligned}
&\partial_\tau w^j + A^i(w^{j-1}, N^{j-1})\, \partial_i w^j + B(w^{j-1}, N^{j-1}, DN^{j-1}) = 0 \quad , \\
&w^j(0) = w_0^j
\end{aligned}
\tag{9a}
$$

$$
L_\tau(w^{j-1}, Dw^{j-1})\, N^j = 1 \quad , \forall_{\tau\in J^{j-1}} \quad ,
\tag{9b}
$$

since the above conditions on w^{j-1}, λ^{j-1} and N^{j-1} ensure the regularity of the equations and the bijectivity of $L_\tau(w^{j-1}, Dw^{j-1})$.
Set

$$
J^j := \{\tau \in J^{j-1} \,|\, |v^j| < 1,\, 0 \neq \lambda^j \geq 0,\, N^j > 0\} \quad ,
$$

then $J^j \supsetneq \{0\}$ holds: The compactness of Σ and $|v_0^j| < 1$, $0 \neq \lambda_0^j \geq 0$ give $|v^j| < 1$, $0 \neq \lambda^j \geq 0$ on a non-empty interval by continuity and $N^j > 0$ follows from the minimum principle for $-L_\tau(w^{j-1}, Dw^{j-1})N^j = -1$. Hence we have constructed the desired sequence of solutions $(w^j, N^j) \in G$ on J^j.

3.2 The energy estimate

Now we are going to investigate the convergence properties of (w^j, N^j). Choose $R = (R_1, R_2)$, $R_1, R_2 > 0$ appropriate, such that

$$\begin{aligned}\tilde{G}_w &:= B_{R_1}(\operatorname{Im} w_0) \subset\subset G_w \\ \tilde{G}_{Dw} &:= B_{R_2}(\operatorname{Im} Dw_0) \subset\subset G_{Dw}\end{aligned}$$

holds. Moreover, there are domains $\{1\} \subset \tilde{G}_N \subset\subset G_N$, $\{0\} \subset \tilde{G}_{DN} \subset\subset G_{DN}$, which depend on R, $||w_0||_{H^s}$. They will be specified later.

After this remarks let us turn to the fundamental energy estimate, which will set us in the position to prove local existence later on (in subsection 3.3).

3.1. Lemma.
Let $s > \frac{3}{2} + 2$.
Then there is a $T > 0$, with $J := [-T, T] \subset J^j\,, \forall_j$, and a $K > 0$, such that for all j and all $\tau \in J$:

$$\text{(a)} \qquad \begin{aligned} ||w^j(\tau)||_{H^s} &\leq K \\ ||N^j(\tau)||_{H^{s+1}} &\leq K \end{aligned}$$

$$\text{(b)} \qquad \operatorname{Im}\left(w^j(\tau), Dw^j(\tau), N^j(\tau), DN^j(\tau)\right) \subset \tilde{G}_w \times \tilde{G}_{Dw} \times \tilde{G}_N \times \tilde{G}_{DN} \quad .$$

Remark.

For the reader familiar with energy estimates for quasilinear symmetric hyperbolic systems I will explain the connection to the standard proofs (see [Ra]). For the hyperbolic part of the equations most elements of the standard proof carry over to the present situation. The difference is due to the coupling to the elliptic equation through the dependence on (N^{j-1}, DN^{j-1}) in (9a) and (w^{j-1}, Dw^{j-1}) in (9b). Assuming $s > \frac{3}{2} + 2$, instead of demanding only $s > \frac{3}{2} + 1$ decouples the estimates and the proof splits into the standard proof for the hyperbolic part and in a part for the elliptic equation.

Proof (induction over j).

For $j = 0$ set $T := T^0 := \frac{1}{2}\operatorname{diam} J^0$.
Since $w^0(\tau) = w_0^0$ we have

$$\|w^0(\tau) - w_0\|_{C^1} = \|w_0^0 - w_0\|_{C^1} \leq C\|w_0^0 - w_0\|_{H^s} \leq C\epsilon$$

by assumption and the Sobolev embedding theorem 2.3. Without loss of generality we can choose the approximating sequence of the initial values, such that the following conditions hold:

$$\epsilon \leq \frac{1}{2}\min\left(\frac{R_1}{C}, \frac{R_2}{C}\right)$$

$$\|w_0^j\|_{H^s}^2 \leq 2\|w_0\|_{H^s}^2 \qquad \text{and} \qquad \|w_0^{j+1} - w_0^j\|_{H^s}^2 \leq \frac{1}{2^j} \qquad \forall_j$$

(we will use the two latter conditions for the local existence proof in 3.3). Statement (b) of the lemma follows then immediately with the remark $N^0 \equiv 1$, $DN^0 \equiv 0$.
Next, we have $\|N^0(\tau)\|_{H^{s+1}} = \|1\|_{H^{s+1}} = \|1\|_{L^2} \leq C$, so, if we set

$$K := K^0 := \max\left(\|w_0\|_{H^s} + \epsilon, \|1\|_{L^2} \right)$$

then (a) follows and we have completed the proof for $j = 0$.

Now we want to perform the induction from $j-1$ to j. This proceeds in four steps.
Let $\tilde{T}$,$\tilde{K}$ be defined for $(w^0, N^0), \ldots, (w^{j-1}, N^{j-1})$.

Step 1 (*Estimate of* $\|w^j(\tau)\|_{H^s}$)
If α denotes a multiindex of order $|\alpha| = s$, then differentiation of (9a) yields

$$\begin{aligned} 0 = \partial_\tau D^\alpha w^j &+ \underbrace{A^i(w^{j-1}, N^{j-1})\, \partial_i D^\alpha w^j}_{\mathrm{I}} \\ &+ \underbrace{\left[D^\alpha \left(A^i(w^{j-1}, N^{j-1})\, \partial_i w^j\right) - A^i(w^{j-1}, N^{j-1})\, D^\alpha \partial_i w^j\right]}_{\mathrm{II}} \\ &+ \underbrace{D^\alpha \left(B(w^{j-1}, N^{j-1}, DN^{j-1})\right)}_{\mathrm{III}} \quad . \end{aligned}$$

Now we have to estimate the marked terms. Note, that for $|\tau| \leq \tilde{T}$ by induction hypothesis all arguments lie in a set whose compact closure is contained in G. Furthermore, the arguments are bounded in the H^s, resp. H^{s+1} norms, independent of τ and j. Then the Moser estimates (prop. 2.4) can be applied and one gets for each $|\tau| \leq \tilde{T}$

$$\begin{aligned} \int_\Sigma < D^\alpha w^j, \mathrm{I} > = \int_\Sigma < D^\alpha w^j, A^i\, \partial_i D^\alpha w^j > &= \int_\Sigma < D^\alpha w^j, (\partial_i A^i) D^\alpha w^j > \\ \leq \|DA\|_{L^\infty} \|w^j\|^2_{H^s} \leq C \|w^j\|^2_{H^s} & \end{aligned}$$

$$\begin{aligned} \|\mathrm{II}\|_{L^2} &\leq C \left(\|D^s A\|_{L^2} \|Dw^j\|_{L^\infty} + \|DA\|_{L^\infty} \|D^{s-1} D w^j\|_{L^2} \right) \\ &\leq C(1 + \tilde{K}) \left(\|w^j\|_{C^1} + \|w^j\|_{H^s} \right) \end{aligned}$$

$$\|\mathrm{III}\|_{L^2} \leq C(1 + \tilde{K}) \quad ,$$

hence

$$\begin{aligned} \frac{d}{d\tau} \int_\Sigma < D^\alpha w^j, D^\alpha w^j > &= 2 \int_\Sigma < D^\alpha w^j, \partial_\tau D^\alpha w^j > \\ &\leq 2 \int_\Sigma < D^\alpha w^j, \mathrm{I} > + 2 \int_\Sigma |D^\alpha w^j| |\mathrm{II} + \mathrm{III}| \\ &\leq C(1 + \tilde{K}) \left(1 + \|w^j\|_{C^1} + \|w^j\|_{H^s} \right) \|w^j\|_{H^s} \quad . \end{aligned}$$

Integration yields

$$\|w^j(\tau)\|^2_{H^s} \leq \|w^j_0\|^2_{H^s} + C(1 + \tilde{K}) \int_0^{|\tau|} \left(1 + \|w^j(t)\|_{C^1} + \|w^j(t)\|_{H^s} \right) \|w^j(t)\|_{H^s}\, dt$$

This estimate remains true, if we replace $|\tau|$ by $\tilde{T}$, $\|w_0^j\|_{H^s}^2$ by $2\|w_0\|_{H^s}$, $\|w^j(t)\|_{C^1}$ by $C\|w^j(t)\|_{H^s}$ (Sobolev embedding theorem, corollary 2.3) and terms of the form $(1+a)^2$ with $C(1+a^2)$, so we get

$$\|w^j(\tau)\|_{H^s}^2 \leq 2\|w_0\|_{H^s}^2 + C(1+\tilde{K}) \int_0^{\tilde{T}} \left(1 + \|w^j(t)\|_{H^s}^2 \right) dt \quad .$$

Gronwall (prop. 2.1) then gives the final estimate

$$\|w^j(\tau)\|_{H^s}^2 \leq \left(1 + 2\|w_0\|_{H^s}^2 \right) e^{C(1+\tilde{K})\tilde{T}} \quad , \qquad \forall_j \, \forall_{|\tau|\leq\tilde{T}} \quad .$$

Set now

$$\begin{aligned} K_w &:= \max\left(K^0, (1+2\|w_0\|_{H^s}^2) e^C \right) \\ \tilde{T}_w &:= \min\left(T^0, \frac{1}{1+K_w} \right) \quad , \end{aligned}$$

then we have $0 < K_w$, $0 < \tilde{T}_w$ and for all j and $|\tau| \leq \tilde{T}_w$ $\|w^j(\tau)\|_{H^s}$ is bounded by K_w, hence we have statement (a) of the lemma with respect to w.

Step 2 (*Boundedness of* $\|w^j(\tau)\|_{C^1}$)
The equation (9a) gives an Sobolev estimate for $\partial_\tau w^j$:

$$\|\partial_\tau w^j\|_{H^{s-1}} \leq \|A^i \partial_i w^j\|_{H^{s-1}} + \|B\|_{H^{s-1}} \leq C(1+K_w)^2 \quad ,$$

since $\|A^i\partial_i w^j\|_{H^{s-1}} \leq C(1+K_w)K_w$ and $\|B\|_{H^{s-1}} \leq C(1+K_w)$. Further, with multiindices β,γ, $|\beta| = s-1$, $|\gamma| = s-2$ the differentiated equation yields

$$\|D^\gamma \partial_\tau D w^j\|_{L^2} = \|\partial_\tau D^\beta w^j\|_{L^2} \leq \|\mathrm{I}\|_{L^2} + \|\mathrm{II}\|_{L^2} + \|\mathrm{III}\|_{L^2} \quad ,$$

with α replaced by β in each of the terms I,II,III defined above. Since we have $s > \frac{3}{2} + 2$, we can estimate II and III as before with s replaced by $s-1$, and for I we have

$$\|\mathrm{I}\|_{L^2} = \int_\Sigma < A^i \partial_i D^\beta w^j, A^i \partial_i D^\beta w^j > \leq C\|A\|_{L^\infty} \|w^j\|_{H^s}^2 \leq C K_w^2 \quad ,$$

and together

$$\|\partial_\tau D w^j\|_{H^{s-2}} \leq C K_w^2 + C(1+K_w)(1+K_w) \leq C(1+K_w)^2 \quad .$$

It follows

$$\begin{aligned} \|w^j(\tau) - w_0\|_{L^\infty} &\leq \|w_0^j - w_0\|_{L^\infty} + \int_0^{|\tau|} \|\partial_\tau w^j(t)\|_{L^\infty} \, dt \\ &\leq \frac{1}{2}\min(R_1, R_2) + C(1+K_w)^2 |\tau| \\ \|D w^j(\tau) - D w_0\|_{L^\infty} &\leq \|D w_0^j - D w_0\|_{L^\infty} + \int_0^{|\tau|} \|\partial_\tau D w^j(t)\|_{L^\infty} \, dt \\ &\leq \frac{1}{2}\min(R_1, R_2) + C(1+K_w)^2 |\tau| \quad . \end{aligned}$$

If we set

$$T_w := \min\left(\tilde{T}_w, \frac{\frac{1}{2}\min(R_1,R_2)}{C(1+K_w)^2} \right) \quad ,$$

then we have $0 < T_w$ and for all $|\tau| \leq T_w$ $\|w^j(\tau) - w_0\|_{C^1}$ is bounded by $\min(R_1, R_2)$, which means, that statement (b) of the lemma is fulfillled with respect to w.

Step 3 (*Estimate of* $\|N^j(\tau)\|_{H^{s+1}}$)
We are looking for a bound of $\|L_\tau(w^{j-1}, Dw^{j-1})^{-1}\|$ in the $H^{s-1} - H^{s+1}$ operator norm. $L_\tau(w^{j-1}, Dw^{j-1})$ is a sum of terms of the form $a_\tau^{j-1} D^\kappa$, with $|\kappa| \leq 2$ and $a_\tau^{j-1} = a_\tau(w^{j-1}, Dw^{j-1})$ where a_τ is a smooth function of its arguments and the parameter τ.

We proceed in several steps.
At first we consider $L_\tau(w^{j-1}, Dw^{j-1})$ as an operator $H^2(\Sigma) \longrightarrow H^0(\Sigma) = L^2(\Sigma)$ only. We already know $\|L_0(w^0, Dw^0)^{-1}\| \leq C$ and show successively

1) $\|L_0(w^j, Dw^j)^{-1}\| \leq C \quad \forall_j$

2) $\|L_\tau(w^j, Dw^j)^{-1}\| \leq C \quad \forall_j \forall_{|\tau| \leq T_N}$, with some $T_N > 0$ chosen appropriately.

The proof of these statements relies on the following standard fact:

Note.

If $T : H^2(\Sigma) \longrightarrow H^0(\Sigma)$ is invertible and $S : H^2(\Sigma) \longrightarrow H^0(\Sigma)$ satisfies

$$\|T - S\| < \|T^{-1}\|^{-1} \quad .$$

Then $S = T - (T - S) = T(1 - T^{-1}(T - S))$ is invertible with norm $\|S^{-1}\| \leq \|T^{-1}\| \sum_{k=0}^\infty q^k = \frac{1}{1-q}\|T^{-1}\|$, where $q = \|T^{-1}(T - S)\| < 1$.

Proof of 1):

$L_0(w, Dw)^{-1}$ exists by assumption. Now define $\delta_j L_0 := L_0(w, Dw) - L_0(w^j, Dw^j)$. We want to show $\|\delta_j L_0\| \leq \epsilon$ for an $0 < \epsilon < \|L_0(w, Dw)^{-1}\|^{-1}$. The note then completes the claim.
Since $w_0^j \longrightarrow w_0$ in $H^s(\Sigma)$ we can choose the sequence such that for the coefficients of $\delta_j L_0$ $\|\delta_j a_0\|_{H^{s-1}} \leq \epsilon$ holds (where the Moser estimates (see prop. 2.4) have been applied on some expression, coming from the mean value theorem, compare the note in subsection 3.3. In particular we have for $f \in H^2(\Sigma)$ the estimate $\|\delta_j L_0 f\|_{H^0} \leq C\|\delta_j a_0\|_{L^\infty}\|f\|_{H^2} \leq C\|f\|_{H^2}$ and we are done.

Proof of 2):

We have shown in 1), that $\|L_0(w^j, Dw^j)^{-1}\| \leq \tilde{C}$, hence $\tilde{C}^{-1} \leq \|L_0(w^j, Dw^j)^{-1}\|^{-1}$. We set $\delta_\tau L(w^j, Dw^j) := L_0(w^j, Dw^j) - L_\tau(w^j, Dw^j)$ and must show, that the estimate $\|\delta_\tau L(w^j, Dw^j)\| < \tilde{C}^{-1}$ holds.
For $f \in H^2(\Sigma)$ and $|\kappa| \leq 2$ we have $\|\delta_\tau a^j D^\kappa f\|_{H^0} \leq C\|\delta_\tau a^j\|_{L^\infty}\|f\|_{H^2}$, and it is enough to find a bound for $\|\delta_\tau a^j\|_{L^\infty}$. For this we mention, that $\delta_\tau a^j = \int_0^\tau \partial_\tau a^j$. Thus

$$\begin{aligned}
\|\delta_\tau a^j\|_{L^\infty} &\leq \|\max_\tau |\partial_\tau a^j|\, \tilde{T}\|_{L^\infty} = \max_x(\max_\tau |\partial_\tau a^j|)\, \tilde{T} \\
&= \max_\tau(\max_x |\partial_\tau a^j|)\, \tilde{T} = \max_\tau \|\partial_\tau a^j\|_{L^\infty}\, \tilde{T} \leq C(1 + K_w)\tilde{T}
\end{aligned}$$

Finally choose some $0 < T_N \leq \tilde{T}$ with $C(1 + K_w)T_N \leq \tilde{C}^{-1}$.

Until now we have proven $\|N^j(\tau)\|_{H^2} \leq C\|L_\tau(w^{j-1}, Dw^{j-1})^{-1}\| \leq C$, $\forall_j$, $\forall_{|\tau| \leq T_N}$, where $L_\tau(w^{j-1}, Dw^{j-1})$ acts as an operator $H^2(\Sigma) \longrightarrow H^0(\Sigma)$. What remains to do is to get

control in higher norms. For this note, that the coefficients a_τ^{j-1} of $L_\tau(w^{j-1}, Dw^{j-1})$ have regularity $\|a_\tau^{j-1}\|_{H^{s-1}} \leq C(1+\tilde{K})$ and therefore $\|a_\tau^{j-1}\|_{C^1} \leq C(1+\tilde{K})$. Now let us consider

$$\begin{aligned} L_\tau(w^{j-1}, Dw^{j-1})(DN^j) &= D[L_\tau(w^{j-1}, Dw^{j-1})N^j] - (DL_\tau(w^{j-1}, Dw^{j-1}))N^j \\ &= -(DL_\tau(w^{j-1}, Dw^{j-1}))N^j \end{aligned}$$

and $\|(Da_\tau^{j-1})D^2N^j(\tau)\|_{L^2} \leq \|Da_\tau^{j-1}\|_{L^\infty}\|N^j(\tau)\|_{H^2} \leq C(1+\tilde{K})$. Therefore, the right-hand side of the equation above lies in $L^2(\Sigma)$ and is bounded there independently of j and τ, $L_\tau(w^{j-1}, Dw^{j-1})$ is an isomorphism $H^2(\Sigma) \longrightarrow L^2(\Sigma)$, thus $DN^j(\tau) \in H^2(\Sigma)$, bounded by

$$\|DN^j(\tau)\|_{H^2} \leq \|L_\tau(w^{j-1}, Dw^{j-1})^{-1}\| \|(DL_\tau(w^{j-1}, Dw^{j-1}))N^j\|_{L^2} \leq C(1+\tilde{K}) \quad ,$$

thus $\|N^j(\tau)\|_{H^3} \leq C(1+\tilde{K})$, and we have control over $N^j(\tau)$ with one differentiability order increased.
Now we iterate this procedure to get estimates on higher norms. For multiindices β, γ, with $|\gamma| = |\beta| - 1$ we find

$$\begin{aligned} &L_\tau(w^{j-1}, Dw^{j-1})(D^\beta N^j) \\ &\quad = D\{L_\tau(w^{j-1}, Dw^{j-1})(D^\gamma N^j)\} - (DL_\tau(w^{j-1}, Dw^{j-1}))D^\gamma N^j \\ &\quad = \sum_{\kappa+\lambda=\gamma} D^\kappa(DL_\tau(w^{j-1}, Dw^{j-1}))D^\lambda N^j \quad , \end{aligned}$$

where successively in each step the previous results has been inserted. Since we have a bound at most for $\|a_\tau^{j-1}\|_{H^{s-1}}$ we get the highest order estimate on $|\beta| = s-1$. Thus we have $D^{s-1}N^j(\tau) \in H^2(\Sigma)$, with $\|N^j(\tau)\|_{H^{s+1}} \leq C(1+\tilde{K})$. Setting $K_N := C(1+\tilde{K})$ completes the proof of (a) of the lemma with respect to N.

Step 4 (*Boundedness of* $\|N^j(\tau)\|_{C^1}$)
Multiplication with the Sobolev constant (compare cor. 2.3) for $H^{s+1} \hookrightarrow C^1$ yields

$$\|N^j(\tau)\|_{C^1} \leq \underbrace{CK_N}_{=:R_N} \quad ,$$

so we can set

$$\tilde{G}_N := B_{R_N}(\{1\}) \quad , \qquad \tilde{G}_{DN} := B_{R_N}(\{0\}) \quad .$$

These domains are independent of τ and j and well defined, $N^j(\tau)$, $DN^j(\tau)$ are contained in $\tilde{G}_N$, $\tilde{G}_{DN}$ for all j,$|\tau| \leq \tilde{T}$, giving statement (b) of the lemma with respect to N and setting

$$\begin{aligned} T &:= \min(T_w, T_N) \leq \tilde{T} \\ K &:= \max(K_w, K_N) \end{aligned}$$

finishes the proof of the lemma. ■

Inspection of the proof shows, that we have shown more. In particular we have the

3.2. Corollary.
With $s > \frac{3}{2} + 2$ *and* $\|N^j(\tau)\|_{H^{s+1}} \leq K_N$ *it follows*

$$\|N^j(\tau)\|_{C^3} \leq R_N \quad ,$$

hence $\mathrm{Im}(D^\alpha N^j) \subset \tilde{G}_N$, *for* $|\alpha| \leq 3$ *and all* j *and* $|\tau| \leq T$.

and

3.3. Corollary.
For all j and $|\tau| \leq T$ the operator $L_\tau(w^j, Dw^j) : H^2(\Sigma) \longrightarrow L^2(\Sigma)$ is invertible and the estimate

$$\|L_\tau(w^j, Dw^j)^{-1}\| \leq C$$

holds.

Moreover, the constants T and K depend only on R and $\|w_0\|_{H^s}$.

3.3 Local existence

The aim of the present subsection is to establish the local in time existence theorem, by proving convergence of the sequence (w^j, N^j) towards a solution of the system (8). Since Σ is compact, we will get spatially global results.

With the notation

$$\delta_j u := u^{j+1} - u^j$$

for any given sequence (u^j) we have to estimate the differences $\delta_j w$ and $\delta_j N$ in Banach spaces of the form

$$\left(C^l(J, X), \|\cdot\|_{C^l(J),X} \right) \quad , \|u\|_{C^l(J),X} := \sum_{i=0}^{l} \sup_{\tau \in J} \|D^i u\|_X \quad , J \subset \tilde{J} := [-\tilde{T}, \tilde{T}] \quad ,$$

where $\tilde{T}$ denotes the time constant of the energy estimate and X an appropriate Sobolev space. Cauchy sequences with respect to this topology are uniformly convergent in τ.

As a consequence of the mean value theorem we

Note.
For $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^l$ let $F : U \times V \longrightarrow \mathbb{R}^m$ a C^1-function.
Then there is a continuous function M on $(U \times V)^2$, with $M \equiv M_v \oplus M_u \in \mathbb{R}^{m \times (k+l)}$, $M_v \in \mathbb{R}^{m \times k}$, $M_u \in \mathbb{R}^{m \times l}$, such that for all $u_1, u_2 \in U$, $v_1, v_2 \in V$

$$F(u_2, v_2) - F(u_1, v_1) = M \begin{pmatrix} u_2 - u_1 \\ v_2 - v_1 \end{pmatrix} = M_v(u_2 - u_1) + M_u(v_2 - v_1)$$

holds.

This kind of replacement will be used frequently in the sequel.

After these preparations the equations for the differences will be derived: Fix $\tau \in \tilde{J}$, then it follows from $0 = \partial_\tau w^j + A^i(w^{j-1}, N^{j-1})\, \partial_i w^j + B(w^{j-1}, N^{j-1}, DN^{j-1}) = \partial_\tau w^{j+1} + A^i(w^j, N^j)\, \partial_i w^{j+1} + B(w^j, N^j, DN^j)$:

$$\begin{aligned} 0 = \partial_\tau \delta_j w + A^i(w^j, N^j)\, \partial_i \delta_j w + \tilde{A}^i(w^j, w^{j-1}, N^j, N^{j-1})\, \partial_i w^j \begin{pmatrix} \delta_{j-1} w \\ \delta_{j-1} N \end{pmatrix} \\ + \tilde{B}(w^j, w^{j-1}, N^j, DN^j, N^{j-1}, DN^{j-1}) \begin{pmatrix} \delta_{j-1} w \\ \delta_{j-1} N \\ \delta_{j-1} DN \end{pmatrix} \quad , \end{aligned}$$

hence the equation fulfilled by the difference $\delta_j w$ is

$$0 = \partial_\tau \delta_j w + A^i(w^j, N^j)\, \partial_i \delta_j w \tag{10a}$$
$$+ B(w^j, Dw^j, w^{j-1}, N^j, DN^j, N^{j-1}, DN^{j-1}) \begin{pmatrix} \delta_{j-1} w \\ \delta_{j-1} N \\ \delta_{j-1} DN \end{pmatrix}$$

and B is a smooth function of its arguments.

For N one gets in the same manner from $1 = L_\tau(w^{j-1}, Dw^{j-1})N^j = L_\tau(w^j, Dw^j)N^{j+1}$:

$$\begin{aligned}
0 &= L_\tau(w^j, Dw^j)N^{j+1} - L_\tau(w^{j-1}, Dw^{j-1})N^j \\
&= L_\tau(w^j, Dw^j)N^{j+1} - L_\tau(w^j, Dw^j)N^j + L_\tau(w^j, Dw^j)N^j - L_\tau(w^{j-1}, Dw^{j-1})N^j \\
&= L_\tau(w^j, Dw^j)\delta_j N + \left(L_\tau(w^j, Dw^j) - L_\tau(w^{j-1}, Dw^{j-1}) \right) N^j \\
&= L_\tau(w^j, Dw^j)\delta_j N - f_\tau(w^j, Dw^j, w^{j-1}, Dw^{j-1}, N^j, DN^j, D^2N^j) \begin{pmatrix} \delta_{j-1} w \\ \delta_{j-1} Dw \end{pmatrix} ,
\end{aligned}$$

hence the equation for $\delta_j N$ is

$$L_\tau(w^j, Dw^j)\delta_j N = f_\tau(w^j, Dw^j, w^{j-1}, Dw^{j-1}, N^j, DN^j, D^2N^j) \begin{pmatrix} \delta_{j-1} w \\ \delta_{j-1} Dw \end{pmatrix} \tag{10b}$$

and f_τ is a smooth function of its arguments and its parameter τ.

Inspection of the last equation yields the important

3.4. Lemma.

Let $s > \frac{3}{2} + 2$.

Then there exists a constant $C > 0$, such that

$$\|\delta_j N(\tau)\|_{H^2} \le C \|\delta_{j-1} w(\tau)\|_{H^1} \quad , \qquad \forall_j \ \forall_{\tau \in \tilde{J}}$$

holds.

The proof of this lemma is an easy consequence of the fact, that the right-hand side of (10b) is an element of $L^2(\Sigma)$ and $L_\tau(w^j, Dw^j) : H^2(\Sigma) \longrightarrow L^2(\Sigma)$ is an isomorphism, with bounded inverse by corollary 3.3.

3.5. Remark.

With this lemma at hand, we achieve again a decoupling of our estimates. Moreover, Cauchy sequences $(\delta_j w)$ in $C^0(\tilde{J}, H^1(\Sigma))$ imply Cauchy sequences $(\delta_j N)$ in $C^0(\tilde{J}, H^2(\Sigma))$. Since $\|\delta_j N(\tau)\|_{H^{s+1}}$ and $\|\delta_j w(\tau)\|_{H^s}$ are uniformly bounded, we get the inequality $\|\delta_j N(\tau)\|_{H^r} \le C \|\delta_{j-1} w(\tau)\|_{H^{r-1}}$ uniformly in $\tau \in \tilde{J}$ for all $1 \le r < s+1$ by interpolation.

Now let us turn to the estimate of the sequence $(\delta_j w)$ in $C^0(\tilde{J}, H^1(\Sigma))$. Applying D on (10a) we infer from the energy estimate 3.1, that all coefficients of the differentiated equation are bounded in all relevant norms (independent of j and $\tau \in \tilde{J}$). So we get the following estimate (in the same way as we have done so far in the energy estimate for equation (9a)).

$$\begin{aligned}
\frac{d}{d\tau} \|D\delta_j w\|^2_{L^2} &= \frac{d}{d\tau} \int_\Sigma < D\delta_j w, D\delta_j w >= 2 \int_\Sigma < D\delta_j w, \partial_\tau D\delta_j w > \\
&\le C \left(\|D\delta_j w\|_{L^2} + \|D\delta_{j-1} w\|_{L^2} + \|D\delta_{j-1} N\|_{H^1} \right) \|D\delta_j w\|_{L^2} \quad ,
\end{aligned}$$

hence

$$\frac{d}{d\tau}\|\delta_j w\|_{H^1}^2 \leq C\left(\|\delta_j w\|_{H^1}^2+\|\delta_{j-1}w\|_{H^1}^2+\|\delta_{j-1}N\|_{H^2}^2\right)$$
$$\leq C\left(\|\delta_j w\|_{H^1}^2+\|\delta_{j-1}w\|_{H^1}^2+\|\delta_{j-2}w\|_{H^1}^2\right) \quad ,$$

where lemma 3.4 has been used in the last step. Integration yields for all $\tau\in\tilde{J}$ (this means $|\tau|\leq\tilde{T}$), using the abbreviation $\delta_j w_0 := w_0^{j+1}-w_0^j$:

$$\|\delta_j w(\tau)\|_{H^1}^2 \leq \|\delta_j w_0\|_{H^1}^2 + C\int_0^{\tilde{T}}\left(\|\delta_j w(t)\|_{H^1}^2+\|\delta_{j-1}w(t)\|_{H^1}^2+\|\delta_{j-2}w(t)\|_{H^1}^2\right)dt$$
$$\leq\left(\|\delta_j w_0\|_{H^1}^2 + C\int_0^{\tilde{T}}\left(\|\delta_{j-1}w(t)\|_{H^1}^2+\|\delta_{j-2}w(t)\|_{H^1}^2\right)dt\right)e^{C\tilde{T}} \quad ,$$

where the Gronwall estimate (prop. 2.1) has been applied in the second step. Since all operations were independent of τ, we can improve the estimate to

$$\|\delta_j w\|_{C^0(\tilde{J}),H^1}^2 \leq \left(\|\delta_j w_0\|_{H^1}^2 + C\tilde{T}\left(\|\delta_{j-1}w\|_{C^0(\tilde{J}),H^1}^2+\|\delta_{j-2}w\|_{C^0(\tilde{J}),H^1}^2\right)\right)e^{C\tilde{T}} .$$

Choose a $T>0$, such that $CTe^{CT}\leq q<1$, and set $J:=[-T,T]\subset\tilde{J}$. Then we get

$$\|\delta_j w\|_{C^0(J),L^2}^2 \leq \|\delta_j w_0\|_{H^1}^2\, e^{CT} + q\left(\|\delta_{j-1}w\|_{C^0(J),H^1}^2+\|\delta_{j-2}w\|_{C^0(J),H^1}^2\right) \quad ,$$

summation over $j=2,\ldots,m$ and rearranging of the terms gives

$$\sum_{j=2}^m\|\delta_j w\|_{C^0(J),H^1}^2$$
$$\leq\frac{1}{1-2q}\left(\left(\sum_{j=2}^m\|\delta_j w_0\|_{H^1}^2\, e^{CT}\right)+q\left(\|\delta_0 w\|_{C^0(J),H^1}^2+2\|\delta_1 w\|_{C^0(J),H^1}^2\right)\right)$$

Clearly, the right-hand side of this inequality is bounded (by choice of the approximating sequence of initial values, we have $\|\delta_j w_0\|_{H^1}^2\leq\frac{1}{2^j}$, see the $j=0$-step in the proof of the energy estimate), independent of m, so we have found a majorant for the left-hand side. Thus the sequence (w^j) is a Cauchy sequence, hence converges in the Banach space $C^0(J,H^1(\Sigma))$. Moreover, standard arguments for the interpolation of Sobolev norms yield convergence in $C^0(J,H^{s'}(\Sigma))$, for all s', with $1<s'<s$, since $\delta_j w$ is bounded in $C^0(J,H^s(\Sigma))$:

$$w:=\lim_{j\to\infty}w^j \quad\text{in}\quad C^0(J,H^{s'}(\Sigma)) \quad ,$$

and the Sobolev embedding theorem asserts, that w is a classical function of regularity $C^0(J,C^2(\Sigma))\subset C^0(J\times\Sigma)$.

It remains to show, that w is a solution of (8a). For this purpose we investigate the sequence $(\partial_\tau\delta_j w)$. Equation (10a) yields

$$\|\partial_\tau\delta_j w\|_{C^0(J),H^1} = \|A^i(w^j,N^j)\,\partial_i\delta_j w + B(\ldots)\begin{pmatrix}\delta_{j-1}w\\ \delta_{j-1}N\\ \delta_{j-1}DN\end{pmatrix}\|_{C^0(J),H^1}$$
$$\leq C\left(\|\delta_j w\|_{C^0(J),H^2}+\|\delta_{j-1}w\|_{C^0(J),H^1}+\|\delta_{j-1}N\|_{C^0(J),H^2}\right)$$
$$\leq C\left(\|\delta_j w\|_{C^0(J),H^2}+\|\delta_{j-1}w\|_{C^0(J),H^1}+\|\delta_{j-2}w\|_{C^0(J),H^1}\right) \quad ,$$

with C again independent of j and τ by the energy estimate. This shows, that $(\partial_\tau w^j)$ is a Cauchy sequence in $C^0(J, H^1(\Sigma))$ and since $\partial_\tau \delta_j w$ is bounded in $C^0(J, H^{s-1}(\Sigma))$ interpolation defines

$$\tilde{w} := \lim_{j\to\infty} \partial_\tau w^j \quad \text{in} \quad C^0(J, H^{s'-1}(\Sigma)) \quad ,$$

and the Sobolev embedding theorem guaranties, that $\tilde{w}$ is a classical function, lying in $C^0(J, C^1(\Sigma)) \subset C^0(J \times \Sigma)$.
The convergence of the sequences (w^j) and $(\partial_\tau w^j)$ is by construnction uniform in τ, hence we have $\tilde{w} = \partial_\tau w$ for $w \in C^0(J, H^{s'}(\Sigma)) \cap C^1(J, H^{s'-1}(\Sigma)) \subset C^1(J \times \Sigma)$. Moreover, $\partial_\tau w$ obeys

$$\partial_\tau w = \tilde{w} = \lim_{j\to\infty} \partial_\tau w^j = \lim_{j\to\infty} \big(-A^i(w^{j-1}, N^{j-1})\, \partial_i w^j - B(w^{j-1}, N^{j-1}, DN^{j-1})\big) \ ,$$

with smooth and therefore $C^0(J, H^{s'-1}(\Sigma))$-continuous A^i, B.
We conclude, that w is a solution of equation (8a), if the sequence (N^j) converges in $C^0(J, H^{s'}(\Sigma))$.

Let us now come to the inspection of the sequence $(\delta_j N)$. Lemma 3.4 and remark 3.5 show $\|\delta_j N\|_{C^0(J), H^{s'+1}} \le C \|\delta_{j-1} w\|_{C^0(J), H^{s'}}$ and it easily follows

$$N := \lim_{j\to\infty} N^j \quad \text{in} \quad C^0(J, H^{s'+1}(\Sigma)) \quad .$$

The smoothness of the coefficients in $L_\tau(w^{j-1}, Dw^{j-1})$, $\tau \in J$ ensures, that for all $f \in H^{s'+1}(\Sigma)$, $L_\tau(w^{j-1}, Dw^{j-1})\, f$ converges in the $H^{s'-1}$ norm to $L_\tau(w, Dw) f$, which means convergence $L_\tau(w^{j-1}, Dw^{j-1}) \stackrel{j\to\infty}{\longrightarrow} L_\tau(w, Dw)$ in the $H^{s'+1} - H^{s'-1}$ operator norm. From this we can infer $L_\tau(w^{j-1}, Dw^{j-1}) N^j \longrightarrow L_\tau(w, Dw) N$ uniform with respect to τ in $H^{s'-1}(\Sigma)$, since

$$\begin{aligned} &\|L_\tau(w, Dw) N - L_\tau(w^{j-1}, Dw^{j-1}) N^j\|_{H^{s'-1}} \\ &\quad \le \|L_\tau(w, Dw) - L_\tau(w^{j-1}, Dw^{j-1})\| \, \|N(\tau)\|_{H^{s'+1}} \\ &\qquad + \|L_\tau(w^{j-1}, Dw^{j-1})\| \, \|N(\tau) - N^j(\tau)\|_{H^{s'+1}} \end{aligned}$$

and both summands consist of one bounded and one converging to zero term. The only possible limit for $L_\tau(w, Dw) N$ is 1, since $L_\tau(w^{j-1}, Dw^{j-1}) N^j = 1$ for all j. Hence N solves equation (8b).
Furthermore, the Sobolev embedding theorem gives regularity $C^3(\Sigma)$ for each $N(\tau)$, and since $w(\tau) \in C^2(\Sigma)$, $Dw(\tau) \in C^1(\Sigma)$, $L_\tau(w, Dw)$ acts as an operator $C^3(\Sigma) \longrightarrow C^1(\Sigma)$, establishing $N(\tau)$ as a classical solution of (8b) for each $\tau \in J$.
Altogether we have proven the following

3.6. Theorem.
Using the conventions described in subsection 3.1 given the system (8) *with spacelike initial values* $w_0 \in H^s(\Sigma)$, $s > \frac{3}{2} + 2$ *and* $0 \ne \lambda_0 \ge 0$, *there exist a non-empty interval* $J = [-T, T]$ *and classical solutions* (w, N) *of* (8), *satisfying*

$$\begin{aligned} w &\in C^0(J, H^{s'}(\Sigma)) \cap C^1(J, H^{s'-1}(\Sigma)) \cap C^1(J \times \Sigma) \\ N &\in C^0(J, H^{s'+1}(\Sigma)) \cap C^0(J, C^3(\Sigma)) \quad , \end{aligned}$$

for each $s' \in \mathbb{R}$ *with* $1 < s' < s$.

3.4 Improving the regularity

Now we want to get rid of the primed s, as well as to investigate the effect of time derivatives on w and N, in order to make the results more satisfying.

Standard arguments on symmetric hyperbolic equations (see [Ra]) show, that $w(\tau)$ possesses indeed regularity H^s, since only the energy estimate and the convergence of the sequence (w^j) are involved in the arguments. For $N(\tau)$ we simply replace $s'+1$ by s and we get

3.7. Note.

$$\begin{aligned} &w \in C^0(J, H^s(\Sigma)) \\ &N \in C^0(J, H^s(\Sigma)) \quad , \end{aligned}$$

so we have a well-defined initial value mapping $w_0 \longmapsto (w(\tau), N(\tau))$, mapping $H^s(\Sigma)$ into itself for all $\tau \in J$.

We want to know something about the effect of differentiation with respect to time. For w^j we have seen, that applying ∂_τ on w^j decreases spatial differentiability by one. For N^j we see first, that in $H^2(\Sigma)$ we get (compare corollary 3.3 and step 3 in the proof of the energy estimate 3.1): $N(\tau) = \lim_j N^j(\tau)$ uniformly in τ, and since $N^j(\tau) \in H^{s+1}(\Sigma)$ this also holds in $H^{s'+1}(\Sigma)$ by interpolation. The same conclusion for the sequence $(\partial_\tau N^j(\tau))$ holds for $s'+1$ replaced by s', since the coefficients of the operator $\partial_\tau L_\tau(w^{j-1}, Dw^{j-1})$ have regularity decreased by one. Thus $(\partial_\tau N^j(\tau))$ converges uniformly with respect to τ in $H^{s'}(\Sigma)$ and this is enough to show, that $\partial_\tau N(\tau) = \lim_j \partial_\tau N^j(\tau)$ in $H^{s'}(\Sigma)$:

3.8. Note.

$$N \in C^0(J, H^{s'+1}(\Sigma)) \cap C^1(J, H^{s'}(\Sigma))$$

Let us combine the previous results. Iteration of the calculations show, that the effect of time differentiating w or N is a decrease of spatial differentiability order by one and we get the

3.9. Corollary.

$$(w, N) \in \bigcap_{l \le s} C^l(J, H^{s-l}(\Sigma)) \quad ,$$

in particular

$$w_0 \in C^\infty(\Sigma) \qquad \Longrightarrow \qquad (w, N) \in C^\infty(J \times \Sigma) \quad .$$

3.5 Uniqueness

Finally we consider the question of uniqueness. Let two classical solutions (w_a, N_a), $a = 1, 2$ of the system (8) with initial value $w_0 \in H^s(\Sigma)$ be given, with regularity

$$\begin{aligned} &w_a \in C^0(J, H^s(\Sigma)) \cap C^1(J, H^{s-1}(\Sigma)) \\ &N_a \in C^0(J, H^{s'+1}(\Sigma)) \end{aligned}$$

for $s > \frac{3}{2} + 2$, $0 < s' < s$.
Define $\mu := w_1 - w_2$, $\nu := N_1 - N_2$, then μ obeys the difference equation

$$\partial_\tau \mu + A^i(w_1, N_1)\, \partial_i \mu + \hat{B}(w_a, Dw_a, N_a, DN_a) \left(\begin{smallmatrix} \mu \\ \nu \\ D\nu \end{smallmatrix} \right) = 0 \quad .$$

This is an inhomogeneous linear symmetric hyperbolic equation of the form $\partial_\tau \mu + A^i \partial_i \mu + B\mu = f$, whose coefficients depend after insertion of the solutions (w_a, N_a) only on (τ, x) and $f(\tau, x) = f(\nu(\tau, x), D\nu(\tau, x))$. The energy estimate for equation of this type yields (see prop. 2.6)

$$\|\mu(\tau)\|^2_{H^{s'}} \leq C \left(\|\mu(0)\|^2_{H^{s'}} + \sup_{0 \leq t \leq \tau} \|f(t)\|^2_{H^{s'}} \right) \quad ,$$

In the same way ν fulfills an analogous difference equation, and can be estimated like

$$\begin{aligned} &\|\nu\|_{H^{s'+1}} \leq C\|L(w_1, Dw_1) - L(w_2, Dw_2)\| \leq C\| \left(\begin{smallmatrix} \mu \\ D\mu \end{smallmatrix} \right) \|_{H^{s'-1}} \leq C\|\mu\|_{H^{s'}} \\ &\|D\nu\|_{H^{s'}} \leq C\| \left(\begin{smallmatrix} \mu \\ D\mu \\ D^2\mu \end{smallmatrix} \right) \|_{H^{s'-2}} \leq C\|\mu\|_{H^{s'}} \quad , \end{aligned}$$

for each τ. This simplifies the energy estimate for μ to

$$\|\mu(\tau)\|^2_{H^{s'}} \leq C \left(\|\mu(0)\|^2_{H^{s'}} + \sup_{0 \leq t \leq \tau} \|\mu(t)\|^2_{H^{s'}} \right) \quad .$$

Since we have $\mu(0) = 0$ by assumption, we find $\mu(\tau) = 0$ for all $\tau \in J$. Furthermore $\|\nu(\tau)\|_{H^{s'+1}} \leq C\|\mu(\tau)\|_{H^{s'}}$ yields $\nu = 0$, hence

$$\begin{aligned} &w_1 = w_2 \\ &N_1 = N_2 \quad , \end{aligned}$$

as desired.

4 Conclusion and outlook

Putting together theorem 3.6, corollary 3.9 and the uniqueness result we get the final

4.1. Theorem.

Let (M, g) be a smooth, globally hyperbolic spacetime, obeying the strong energy condition, with compact Cauchy surface Σ and

$$\lambda = |k|^2 + 4\pi(\rho + \operatorname{tr} S)$$

does not vanish identically on Σ.
Then there exists a $T > 0$ and a unique smooth PMC foliation $\{S_\tau\}$, $\tau \in [-T, T]$ in (M, g), with $\Sigma = S_0$.

Note, that the setting here is quite general, no symmetry assumptions have to be made and essentially the strong energy condition turned out to be sufficient for the local in time existence of a unique PMC foliation up to the choice of an initial Cauchy surface.

What remains to do is to globalize the result. Here are two problems involved: How large is the interval of values taken by the time coordinate and does the global foliation then cover the whole spacetime? To answer these questions in general there seem to be no techniques available up to now. One strategy to obtain global results is, to study first spacetimes with some spatial symmetry, taking advantage of the simplifications of the equations. Then the hope is, that the techniques developed in these cases give insight into the nature of more general classes of spacetimes, by successively lowering the degree of symmetry. This line of attack will be the content of a forthcoming paper.

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